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Slow interaction dynamics in spin-glass models

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Abstract. The spin model of Coolen *et al*, involving slow dynamical laws for the couplings linking fast spins, is considered as a spin-glass model having explicit quenched disorder. The thermodynamic behaviour predicted is reminiscent of a Sherrington–Kirkpatrick spin-glass with one-step replica-symmetry breaking—generalizing the parallels of the simplest form of this model, which was entirely free of frozen disorder. However, even though the slow evolution of the couplings allows some balancing between achieving unfrustrated spin configurations and adopting those favoured by the quenched disorder, it is seen that this is not generally sufficient to avoid replica-symmetry breaking throughout the frozen phases. Moreover, there are seen to be two distinct types of spin-glass phase, each of which has both ergodic (replica-symmetric) and ergodicity-broken regimes.

1. Introduction

The magnetic-spin model introduced by Coolen *et al* (1993) provides a convenient handle on the equilibrium properties of a class of idealized systems in which both spin moments and their mutual interactions may evolve stochastically, on their respective, disparate timescales. Moreover, the simplest form of the model was shown to exhibit close parallels with the Sherrington–Kirkpatrick (1975) (SK) spin-glass model, even in the absence of explicit quenched disorder, and also to offer a novel perspective on the replica method. In this paper we will examine the effects of including an explicit frozen randomness into the dynamics which directly induces the interaction weights to have a similar form to those of the SK model, in which they are fixed disordered variables, while still allowing them some diffusive motion. We will see that this imposition leads to new thermodynamic phases and extends the parallels between the replica method and the original formulation of the model.

We will first outline the dynamical laws that we will be considering, in a slightly more general form than has been presented previously. Thereafter, in section 3 the basic analysis of the model will be discussed, and its relation to the pattern of replica symmetry breaking seen in the SK model will be indicated in section 4. The thermodynamics of the model will be considered in section 5, followed by a discussion of the problems of simulating the model numerically. Our conclusions are offered in section 7.

2. Overview of the model

A brief description of the dynamical laws implicit in the Coolen model will be given; a fuller discussion may be found in Penney *et al* (1993). A set of N Ising spins, $S_i \in \{\pm 1\}$, are imagined to be joined by symmetric, real-valued interaction weights J_{ij}, with both these

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sets of quantities allowed to evolve in time. Centrally, it is assumed that the (stochastic) spin dynamics are very fast compared with those of their interactions, so that on the timescale over which these couplings change, the spins can be considered always to be in a state of dynamic equilibrium. With the imposition that $J_{ij} = J_{ji}$, the dynamical laws for the spins may be assumed to be such as to give an equilibrium distribution of the Boltzmann form. This will depend on a characteristic temperature of the spin dynamics β^{-1} and a Hamiltonian such as

$$H = -\sum_{i<j} c_{ij} J_{ij} S_i S_j - \sum_i h_i S_i \tag{2.1}$$

in which appear the couplings $\{J_{ij}\}$, along with some externally imposed fields h_i. In general, the connectivity of the system need not be complete, and each parameter $c_{ij} \in \{0, 1\}$ controls whether the bond linking given sites i and j is present. Inspired by processes believed to occur in neurophysiological tissue, as well as by analytic simplicity, the following form of interaction dynamics has been proposed:

$$\tau \frac{\mathrm{d}}{\mathrm{d}t} J_{ij} = \frac{c_{ij}\langle S_i S_j\rangle + K_{ij}}{N^\zeta} - \mu J_{ij} + \frac{\eta_{ij}(t)}{N^{\zeta/2}} \tag{2.2}$$

where $\langle S_i S_j\rangle$ embodies the mutual correlation of spins at either end of the bond J_{ij}, involving a thermal average with respect to the spin dynamics. (The precise form of the evolution of bonds where $c_{ij} = 0$ is of little significance.) External biases $\{K_{ij}\}$ may be used to steer the weights towards some preferred values and uncorrelated Gaussian white noises, $\eta_{ij}(t)$, with an associated temperature scale $\tilde{\beta}^{-1}$, are also involved, producing a Langevin dynamics for the couplings. (The noise is completely specified by its first two moments; $\langle \eta_{ij}(t)\rangle_\eta = 0$, $\langle \eta_{ij}(t)\eta_{kl}(t')\rangle_\eta = 2\tilde{\beta}^{-1}\tau\delta_{(ij)(kl)}\delta(t-t')$.) Factors of N in (2.2) are needed to ensure a sensible thermodynamic limit $N \to \infty$, and depend on the mean connectivity of the network via the quantity ζ,

$$\zeta = \lim_{N\to\infty} \frac{\ln(N\langle c\rangle_c)}{\ln N} \tag{2.3}$$

so that $\zeta = 1$ for a fully-connected system, and $\zeta = 0$ for a network of finite connectivity, such as a finite-dimensional lattice-based model.

With the assumption that the weight dynamics are slow, the spin-correlation functions may be specified in terms of the partition function of the spin system (for a given choice of couplings), and (2.2) may thereby be rewritten as a diffusion in a potential

$$\tau \frac{\mathrm{d}J_{ij}}{\mathrm{d}t} = -\frac{\partial}{\partial J_{ij}} \left\{ -\frac{1}{\beta N^\zeta} \ln Z_\beta - \frac{1}{N^\zeta} \sum_{i<j} J_{ij} K_{ij} + \frac{1}{2}\mu \sum_{i<j} (J_{ij})^2 \right\} + \frac{1}{N^{\zeta/2}} \eta_{ij}(t) \tag{2.4}$$

where the spin-partition function is given by the standard expression

$$Z_\beta(\{J_{ij}\}) = \mathrm{Tr}_{\{S_i\}} \exp(-\beta H(\{S_i\}, \{J_{ij}\})) . \tag{2.5}$$

This identification allows the equilibrium distribution of the couplings to be specified immediately, it being another Boltzmann distribution, with an associated partition function of the form

$$\tilde{Z}_{\tilde{\beta}} = \int \prod_{i<j} \left\{ \frac{\tilde{\beta}\mu N^\zeta}{2\pi} \right\}^{1/2} \mathrm{d}J_{ij}\ [Z_\beta]^n \exp\left\{ \tilde{\beta} \sum_{i<j} J_{ij} K_{ij} - \frac{1}{2}\tilde{\beta}\mu N^\zeta \sum_{i<j} (J_{ij})^2 \right\} \qquad n = \tilde{\beta}/\beta . \tag{2.6}$$

The ratio of characteristic (positive) temperatures, denoted by n, will later become identified with the dimensionality of a replica space. From $\ln \tilde{Z}_{\tilde{\beta}}$ one may extract cumulant averages of the weights, along with appropriate averages of the spin moments, by differentiation with respect to h_i and K_{ij}, interpreting these as sources in a field-theoretic sense.

In the basic formulation of this model, both these sets of sources were taken to be homogeneous ($h_i = h \ \forall i$, $K_{ij} = K \ \forall i, j$), and complete connectivity was assumed ($c_{ij} = 1 \ \forall i, j$). The model was therefore entirely free of quenched disorder and yet, particularly for small n, exhibited spin-glass character owing to the stochasticity of the weight dynamics and their lethargy with respect to those of the spins. Moreover, in this formulation the model shows close parallels with the Sherrington–Kirkpatrick (SK) spin-glass model which explicitly includes quenched random interactions, although the flexibility of the weights is found to show non-trivial effects in terms of varied phase-transition order and relocation of phase boundaries. In this work we will examine the inclusion of explicit disorder into the model, which in general may be motivated by at least two differing considerations.

Firstly, it would be interesting to examine the effects of assuming the K_{ij} to be quenched disorder variables. The disorder would then merely be a bias on the weight dynamics, rather than being a prescription for the weights themselves. This might parallel a quenched magnetic alloy in which the magnetic ions are relatively firmly fixed at irregular positions (whose mutual influences are represented by the quenched random K_{ij}s), but about which some irregular motion might be possible on slow timescales (reflected in the J_{ij}-dynamics themselves).

An alternative source of quenched disorder can arise in moving towards a finite-dimensional model. Even though the mean-field theory of disordered systems is quite subtle (e.g. Mézard *et al* 1987), the comparative simplifications it offers relative to techniques needed for finite-range systems mean that it is highly convenient to have couplings that are not influenced by the spatial separation of the sites they connect. Therefore, of the two central properties of a low-dimensional lattice-based model, namely short-ranged bonds and finite connectivity, the latter is probably the easier to incorporate. Comparison of the analyses of the Viana–Bray (1985) and Sherrington–Kirkpatrick (1975) spin-glass models shows that even this step is quite cumbersome. Preliminary investigations suggest that finite connectivity introduces few qualitative effects that are not already present in the basic model or the manifestation that we are about to discuss. Most notably, both the disordered models exhibit two distinct types of spin-glass phase, with variable phase transition orders. However, in systems of random finite connectivity, owing to the existence of non-trivial correlations between pairs of couplings the weight-dynamics impact upon the location of phase boundaries over the entire range of temperature ratios n, rather than only for n beyond certain thresholds. There is also a heightened ground-state degeneracy resulting from the presence of extensive numbers of small disconnected spin-clusters.

In a disordered system, the expectation that physical observables should be self-averaging (i.e. insensitive to the precise values of the random variables, and influenced only by their statistical properties) means that it is the generator of such observables $\ln \tilde{Z}_{\tilde{\beta}}$, rather than $\tilde{Z}_{\tilde{\beta}}$ itself, that is anticipated as being self-averaging. The infamous awkwardness of needing to perform the formal average $\langle \ln \tilde{Z} \rangle$ will be tackled by invoking the replica method, via which $\langle \ln \tilde{Z} \rangle$ is replaced by $\lim_{r \to 0}(\langle \tilde{Z}^r \rangle - 1)/r$ and this latter quantity is evaluated by attempting an analytic continuation from integer r. Given that the factor of Z_{β}^n within $\tilde{Z}_{\tilde{\beta}}$ (2.6) will itself require similar continuations for its evaluation, by including disorder in our system two levels of replicas will be needed. This feature leads to parallels with the form of replica-symmetry breaking believed to be exhibited by ordinary disordered systems, and accordingly involves an augmented set of order parameters as compared with the respective canonical systems. For notational conciseness, those r replicas introduced to perform the disorder average will be labelled by indices in Roman script and be called 'Roman' replicas, while those used in treating $[Z_\beta]^n$ will be given Greek-script indices, and be referred to as 'Greek' replicas. Thus, for each of the r replicas involved in the disorder average, there are n replicas related to the temperature ratio $\tilde{\beta}/\beta$, whose labelling is entirely arbitrary with respect to that in other Roman replicas.

3. Quenched random biases

We will confine attention to a fully connected model ($c_{ij} = 1\ \forall i, j$, so that $\zeta = 1$) in which the biases present in the weight dynamics are quenched random variables. For simplicity, these will be assumed to be drawn from independent Gaussian distributions whose mean and variance scale appropriately with system size so as to ensure a sensible thermodynamic limit. Thus the biases will be taken to be distributed according to

$$p(B_{ij}) = \frac{1}{\sqrt{2\pi\tilde{B}/N}} \exp\left\{-\frac{\left(B_{ij} - B_0/N\right)^2}{2\tilde{B}/N}\right\} \tag{3.1}$$

where $B_{ij} = K_{ij}/(\mu N)$, a form closely reminiscent of the bond distribution of the SK model, and henceforth we will assume there to be no external fields on the system, so that $h_i = 0$. Introducing two levels of replicas, one to assist with averaging the free energy of the weight system, and the other to aid evaluation of the Hamiltonian of the couplings, calculation of this free energy begins from

$$\left\langle\left[\tilde{Z}_{\tilde{\beta}}\right]^r\right\rangle_B = \left\langle\int\prod_{a=1}^{r}\left\{\prod_{i<j}\frac{\sqrt{N}\mathrm{d}J_{ij}^a}{\sqrt{2\pi\tilde{J}}}\right\}\exp\left\{-\frac{N}{2\tilde{J}}\sum_{\substack{i<j\\a}}\left(J_{ij}^a - B_{ij}\right)^2\right\}\right.$$
$$\left.\times\left\{\mathrm{Tr}_{\{S_i^{\alpha a}\}}\exp\beta\left[\sum_{\substack{i<j\\\alpha a}} S_i^{\alpha a} J_{ij}^a S_j^{\alpha a}\right]\right\}\right\rangle_B \tag{3.2}$$

in which $\alpha \in \{1, \ldots n\}$, $\tilde{J} = (\tilde{\beta}\mu)^{-1}$ and $\langle\ \rangle_B$ denotes an average over the distributions $p(B_{ij})$. By performing the integrals over the replicated couplings J_{ij}^a, then the biases B_{ij}, and using various Hubbard–Stratonovich transformations, one may reduce (3.2) to an extremization problem in the space of some appropriate order parameters (cf Kirkpatrick and Sherrington 1978). These parameters, in natural generalization of those familiar from typical replica calculations, carry two sets of replica labels. Parameters that involve a pair of replicas can connect either two distinct Greek replicas within a single Roman replica ($q_a^{\alpha\beta}$), or two Greek replicas within distinct Roman replicas ($q_{ab}^{\alpha\beta}$). The precursor of the free energy is thus given by

$$\left\langle\left[\tilde{Z}_{\tilde{\beta}}\right]^r\right\rangle_B = \exp N\left\{\tfrac{1}{4}nr\beta^2(\tilde{B}+\tilde{J}) + nr\ln 2 + \mathrm{extr}_{\{m_\cdot^\cdot\},\{q_{\cdot\cdot}^{\cdot\cdot}\},\{q_\cdot^{\cdot\cdot}\}} F\left(\{m_\cdot^\cdot\},\{q_{\cdot\cdot}^{\cdot\cdot}\},\{q_\cdot^{\cdot\cdot}\};n\right)\right\} \tag{3.3}$$

in which

$$F\left(\{m_\cdot^\cdot\},\{q_{\cdot\cdot}^{\cdot\cdot}\},\{q_\cdot^{\cdot\cdot}\};n\right) = -\sum_{\alpha a}\tfrac{1}{2}\beta B_0 m_a^{\alpha 2} - \sum_{\substack{a<b\\\alpha\beta}}\tfrac{1}{2}\beta^2\tilde{B}q_{ab}^{\alpha\beta 2} - \sum_{\substack{a\\\alpha<\beta}}\tfrac{1}{2}\beta^2(\tilde{B}+\tilde{J})q_a^{\alpha\beta 2}$$
$$+\,G\left(\{m_\cdot^\cdot\},\{q_{\cdot\cdot}^{\cdot\cdot}\},\{q_\cdot^{\cdot\cdot}\};n\right)$$

and

$$G(\{m_\cdot^\cdot\},\{q_{\cdot\cdot}^{\cdot\cdot}\},\{q_\cdot^{\cdot\cdot}\};n)=\ln\left[\frac{1}{2^{nr}}\mathrm{Tr}_{\{S_\alpha^a\}}\exp\left\{\sum_{\substack{a\\ \alpha}}\beta B_0 m_a^\alpha S_\alpha^a+\sum_{\substack{a<b\\ \alpha\beta}}\beta^2\tilde{B}q_{ab}^{\alpha\beta}S_\alpha^a S_\beta^b\right.\right.$$
$$\left.\left.+\sum_{\substack{a\\ \alpha<\beta}}\beta^2(\tilde{B}+\tilde{J})q_a^{\alpha\beta}S_\alpha^a S_\beta^b\right\}\right]. \tag{3.4}$$

The scales of β, B_0 and $\tilde{J}$ may be fixed by setting $\tilde{B}=1$, thereby eliminating an invariance of the dynamical laws in the form originally specified.

It is usual, by way of a first step, to assume that the replica-dependent order parameters that extremize the relevant free-energy functional are independent of their indices. Conventionally, this hypothesis is taken to be equivalent to an assumption of ergodic dynamics. Within this ansatz one may readily perform the analytic continuation $r\to 0$, yielding

$$\lim_{r\to 0}\frac{1}{r}F(m,q_{11},q_2;n)=-n\frac{1}{2}\beta B_0 m^2-\frac{1}{2}n(n-1)\frac{1}{2}\beta^2(1+\tilde{J})(q_2)^2-n^2\frac{1}{2}\beta^2(q_{11})^2$$
$$+\lim_{r\to 0}\frac{1}{r}G(m,q_{11},q_2;n) \tag{3.5}$$
$$\lim_{r\to 0}\frac{1}{r}G(m,q_{11},q_2;n)=\int\mathrm{D}x\,\ln\left[\int\mathrm{D}y\,\cosh^n(\Xi)\right]-\frac{1}{2}n\beta^2(1+\tilde{J})q_2$$

in which

$$\Xi=\beta\left\{B_0 m+x\sqrt{q_{11}}+y\sqrt{(1+\tilde{J})q_2-q_{11}}\right\} \tag{3.6}$$

and moreover give simple interpretations of the order parameters in terms of appropriate averages over the various dynamics,

$$m_a^\alpha=m=\left\langle\overline{\langle S_i\rangle}\right\rangle_B\qquad q_{ab}^{\alpha\beta}=q_{11}=\left\langle\overline{\langle S_i\rangle}^2\right\rangle_B\qquad q_a^{\alpha\beta}=q_2=\left\langle\overline{\langle S_i\rangle^2}\right\rangle_B. \tag{3.7}$$

Herein $\langle\ \rangle$ represents a thermal average over the spin dynamics, for a given choice of couplings and biases, $\overline{}$ is a thermal average over the coupling dynamics with fixed biases, and $\langle\ \rangle_B$ is again an average over the disorder of the bias distribution. The assumption of self-averaging means that an average over sites i may be replaced by a mean with respect to the distribution of the disorder variables. One may give concise expressions for various spin averages in terms of these fundamental order parameters:

$$\left\langle\prod_{a=1}^{r}\overline{\langle S_i\rangle^{\nu_a}}\right\rangle_B=\int\mathrm{D}x\,\prod_{a=1}^{r}\frac{\int\mathrm{D}y\,\cosh^n(\Xi)\tanh^{\nu_a}(\Xi)}{\int\mathrm{D}y\,\cosh^n(\Xi)}. \tag{3.8}$$

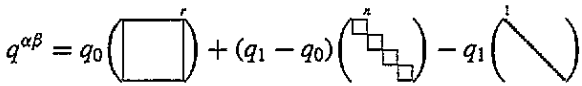
So, for example, m, q_{11} and q_2 correspond to choices $\nu_a=(1,0,\ldots,0)$, $\nu_a=(1,1,0,\ldots,0)$ and $\nu_a=(2,0,\ldots,0)$, respectively. These relations readily yield the inequalities $q_2\geqslant q_{11}\geqslant m^2$.

4. Parallels with replica-symmetry breaking

It is noteworthy that within the replica-symmetric ansatz, the form of the free energy (3.5) is quite reminiscent of the free energy of the SK model within the one-step replica-symmetry breaking approximation:

$$\begin{aligned}
\lim_{r\to 0}\frac{1}{r}F^1_{\mathrm{SK}}(m,q_0,q_1;n) &= -\frac{1}{2}\beta J_0 m^2 - \frac{1}{2}(n-1)\frac{1}{2}\beta^2(q_1)^2 - n\frac{1}{2}\beta^2(q_0)^2 \\
&\quad + \frac{1}{n}\lim_{r\to 0}\frac{1}{r}G^1_{\mathrm{SK}}(m,q_0,q_1;n) \\
\lim_{r\to 0}\frac{1}{r}G^1_{\mathrm{SK}}(m,q_0,q_1;n) &= \int \mathrm{D}x\,\ln\left[\int \mathrm{D}y\,\cosh^n(\widehat{\Xi})\right] - \frac{1}{2}\beta^2 q_1 \\
\widehat{\Xi} &= \beta\left\{J_0 m + x\sqrt{q_0} + y\sqrt{q_1-q_0}\right\}
\end{aligned} \tag{4.1}$$

in which q_1, q_0 and n parameterize the first step in the Parisi ansatz for the order parameter matrix $q^{\alpha\beta}$ (Parisi 1980b) and the replica dimension, here denoted by r, has been taken to zero. We note that our labelling of these parameters differs from that of Parisi (1980a, b) and Mézard *et al* (1987)—these authors use n for the replica dimension, m to describe the geometry of the matrix $q^{\alpha\beta}$, and typically avoid introducing an order parameter to quantify net magnetization—but some conflict with either this notation or that of our original discussion of the coupled dynamical model (Penney *et al* 1993) seems inevitable. As an aid to the visualisation of the parameters as they will be used here, we may depict the one-step Parisi ansatz used in the SK model as follows:


(4.2)

where each block encloses matrix elements that are equal to unity.

In the context of neural network models, it has also been noted that two levels of replicas lead to formulae that exhibit similarities with this pattern of replica symmetry breaking in a single set of replicas (Monasson and O'Kane 1994). The main differences relative to (3.5) are the matching of prefactors of q_1 and q_0 (as compared to $(1+\tilde{J})q_2$ and q_{11}), an overall factor of $1/n$ by which these two functionals differ, and the need to extremize F^1_{SK} with respect to n in addition to m, q_0 and q_1. Dynamically, the breaking of replica symmetry in the SK spin glass is interpreted as resulting from the spin dynamics becoming confined in one of a set of inequivalent thermodynamic states, so that partial freezing occurs, without equilibrium being established between these pure states. Various timescales are then associated with the traversal of the multifarious energy barriers that enclose ergodic components of the spin phase space. The one-step breaking in the SK model therefore parallels the two disparate timescales that are represented by the spin and weight dynamics in the coupled-dynamical formulation†. The limit $\tilde{J}\to 0$ (via $\mu\to\infty$), in which the two qs enter on equal footing, represents a situation in which the biases B_{ij} entirely dictate the states of the couplings, overruling the influence of the spins or Langevin noises. That this

† This suggests that further steps in the Parisi hierarchy of replica-symmetry breaking may be emulated by allowing the biases B_{ij} to have their own, still slower, stochastic dynamics, which in turn are driven by their own very slowly changing biases, and so on. This hierarchy of dynamics is also analogous to the many spin timescales used in Sompolinsky's dynamical analysis of the SK model (1981).

physically corresponds with the SK model itself suggests that ergodicity will not maintain in the present model in this limit. However, it is conceivable that, for finite $\tilde{J}$, this system will be ergodic (in the sense of preserving replica symmetry) and yet show formal similarity with a spin system where ergodicity is broken, and thus that the slow evolution of the weights provides a means of penetrating the energy barriers that obstruct the spin dynamics in the SK model. Indeed, analytically, one means proposed for regulating the long-time behaviour of the SK model is allowing slow evolution of the disordered bond variables (Horner 1984).

Thus the coupled-dynamical model can offer an alternative perspective on the replica method even when replica symmetry is observed to break. Moreover, the present formulation affords an interesting interpretation of the saddle-point condition for the Parisi order parameter n (or m in the notation of Parisi 1980a, b). For the SK model, in the Parisi hierarchical replica-symmetry breaking scheme (Parisi 1980a) the matrix $q^{\alpha\beta}$ is parameterized by a set of spin-averages, $\{q_a\}$, and a set of weight-factors, $\{n_a\}$. The need to extremize the relevant free-energy functional in the mean-field approximation (analogous to (3.4)) with respect to $q^{\alpha\beta}$, translates naturally into an extremization with respect to $\{q_a\}$. Furthermore, it is reasonably proposed (Mézard *et al* 1987) that the variables $\{n_a\}$ should also be varied so as to extremize the free energy. In the coupled dynamical model, where n translates into a physically-significant ratio of characteristic temperatures, one may seek a physical interpretation of the saddle-point condition on n, by considering the implication of the equation

$$\frac{\partial}{\partial n}\frac{F_{\mathrm{extr}}}{n} = 0 \tag{4.3}$$

where F_{extr} is the result of the extremization required in (3.3). This may be translated into the following equality of time-averages:

$$\left(\overline{-\beta^{-1}\ln Z_\beta}\right)_B = \left(-\tilde{\beta}^{-1}\ln\tilde{Z}_{\tilde{\beta}}\right)_B \tag{4.4}$$

i.e. the temperature of the coupling system should be chosen so that the free energy appropriate to the timescale of the spins $(-\beta^{-1}\ln Z_\beta)$ fluctuates symmetrically about the free energy for long timescales, namely that of the couplings themselves $(-\tilde{\beta}^{-1}\ln\tilde{Z}_{\tilde{\beta}})$. (It should be noted that this result does depend on the choice of zero of each free energy; particularly, we have chosen the free energy of the couplings to be zero when the temperature ratio n is zero, under which circumstances the couplings diffuse entirely uninfluenced by the spins.) Given that this consequence of (4.3) is true even in the limit $\tilde{J} \to 0$, representing an SK model, and moreover does not itself depend on the validity of the replica-symmetric assumption in the coupled-dynamical model, the extremality condition $\partial/\partial n\, F^1_{\mathrm{SK}} = 0$ is seen to have an appealing physical analogy. Although this equality of free energies is reminiscent of an equilibrium between two phases of a thermodynamic fluid, it is not clear how to make this simile more definite. However, we will shortly see that achieving the balance between these two free energies (4.4) will itself lead to replica-symmetry breaking even though the coupling dynamics may act to reduce the level of frustration within the spin system.

5. Intrinsic features

Important to the similarities highlighted above is the assumption that the replica-symmetric ansatz is valid for the coupled-dynamical model. It is well known from the studies of the SK model that spin-glass phases and complex thermodynamic states are closely linked, and that whilst RSB need not alter all aspects of qualitative behaviour, analytic descriptions or quantitative characteristics can be changed significantly. Therefore, it is prudent to test the viability of the replica-symmetric approximation to the extremum of the functional (3.4), an ansatz that can fail in at least two distinct ways. Firstly, central to the validity of the saddle-point integration that underlies the extremization, is the local stability of the extremum; i.e. all eigenvalues (in the full replica space) of the full free-energy functional evaluated at the replica-symmetric saddle-point must be such as to ensure convergence of the integrals. Alternatively, even if this condition holds, it is conceivable that a replica-symmetry-broken saddle-point will be the true thermodynamically favoured configuration, such that a first-order transition from a replica-symmetric state to one of broken replica symmetry could occur. Although there are instances where the latter pathology is relevant (e.g. Krauth and Mézard 1989), studies of the canonical realization of the coupled dynamical model suggest that it is failure of the local stability of the replica-symmetric saddle-point that signals the onset of replica-symmetry breaking, rather than any form of discontinuous transition. Following de Almeida and Thouless (1978) we have therefore examined the eigenvalues of the free-energy functional (3.4) at the RS saddle-point; these may be calculated in analogy with those appropriate to the SK model. Given that the full Parisi hierarchy of RSB is believed to be only marginally stable for the SK model in its spin-glass phase, i.e. always to have zero-modes about the Parisi saddle point (de Dominicis and Kondor 1983), it should be emphasized that there is little point in examining the local stability of a one-step breaking scheme for this model itself, for as soon as the replica-symmetric ansatz fails, it is to be expected that only the full hierarchy will avoid instability. However, that the coupled-dynamical model remains inequivalent to SK model until $\tilde{J} \to 0$ means that the similarity of its replica-symmetric formulation to that of the SK model (allowing for one-step RSB) does not necessarily imply that the replica-symmetric assumption is always inadequate for the present model in the spin-glass phases.

The stability of the replica-symmetric saddle-point may be analysed following the method of de Almeida and Thouless (1978); a brief summary of this method is given in appendix A. Given the two levels of replicas present in the analysis, two distinct forms of RSB can occur, signalled by the replicon eigenvalues λ_G and λ_R. Before relating RSB to the phase structure of the model, we note that there do indeed appear to be regions of the frozen phases where the replica-symmetric approximation for the coupled-dynamical model is correct.

Turning now to the phase boundaries themselves, whilst q_{11} and q_2 both resemble the Edwards–Anderson order parameter (Edwards and Anderson 1975) familiar from studies of spin glasses, it should be emphasised that these quantities will play distinct roles in determining the phase structure of the model. Neither parameter is sufficient, solely in combination with m, for distinguishing all types of ordered phases from the high-temperature paramagnetic phases. Given that mean-field theory has the useful habit of implicitly identifying physically significant order parameters, the emergence of q_{11} and q_2 already suggests that two forms of spin-glass phase are to be expected. Numerical extremization of the free-energy functional confirms this distinction, and some typical evolutions of these order parameters in temperature are shown in figure 1.

By expanding the extremality conditions appropriate to (3.4) in terms of the order parameters, one may distinguish first-order phase transitions from second order, and furthermore, where phase changes are second order this method will also yield algebraic conditions for the temperature where these changes occur. In the interests of simplifying this goal, we will hereafter take $B_0 = 0$, so that $m = 0$, indicating a general absence of ferromagnetism. Notationally, paramagnetic phases (where $q_2 = q_{11} = 0$) will be labelled

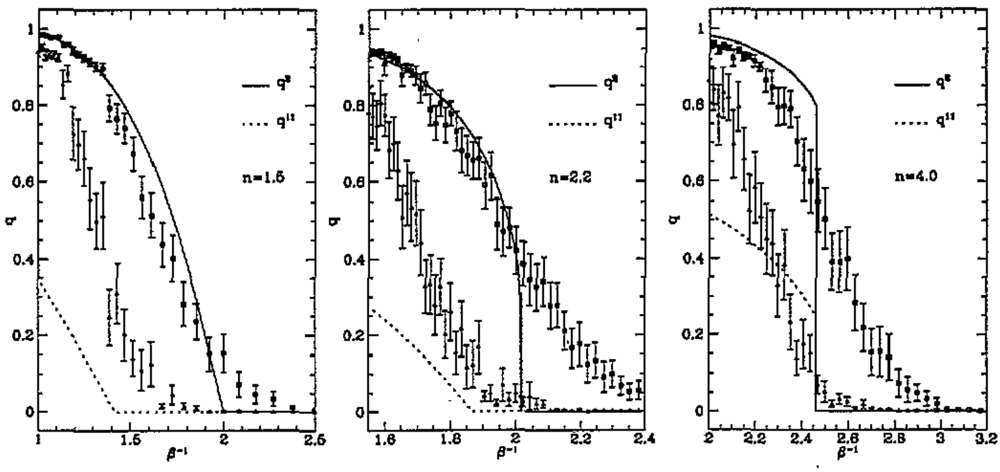


Figure 1. Evolution of order parameters q_2 and q_{11} with spin temperature β^{-1} for various temperature ratios n, all at $B_0 = 0$, $\tilde{B} = 1$, $\tilde{J} = 3$. Also shown are results of computer simulations of the dynamical model; their limited similarity to the analytic predictions is discussed in the text.

'P', those spin-glass-like regions where $q_2 > 0$ and $q_{11} = 0$ will be 'SG1', and situations in which both q_2 and q_{11} are finite will be denoted 'SG2'. Regarding the P$\rightarrow$SG1 transition, this may be shown to be second order for $n < 2$, and occurs where $\beta^2(1 + \tilde{J}) = 1$, whilst if $n > 2$ this boundary is first order and moves towards higher temperature. Where the mutation from SG1 to the more frozen state of SG2 is distinct from the P$\rightarrow$SG1 transition, the former boundary is always second order and occurs for $\beta\{(n-1)q_2+1\} = 1$. However, there is also the possibility of a joint first order transition if $n > 2$, thereby excluding any SG1 behaviour. Numerical solution of the relevant boundary conditions, for a particular choice of $\tilde{J}$, leads to a phase diagram as shown in figure 2. Qualitatively, reducing $\tilde{J}$ moves the P–SG1 transition down towards $1/\beta = 1$ and reduces the range of n beyond 2 over which this transition is distinct from the SG1–SG2 boundary.

The curves defined by the vanishing of the eigenvalues λ_R and λ_G are seen to indicate that replica-symmetry can be lost within the interior of the SG1 phase for $n < 1$, but that this pathology does not occur outside the more frozen SG2 phase while $n > 1$. The Roman AT line (on which $\lambda_R = 0$) crosses the SG1–SG2 at $n = 1$, $\beta = 1$, where there is cusp, while the Greek AT line (where $\lambda_G = 0$) has a form similar to that of the basic coupled-dynamical model, here also signalling re-entrant RSB only beneath $n \simeq 0.3$. However, calculation of the boundary defined by the matching of the spin and coupling free energies within the replica-symmetric ansatz (4.4) indicates that this lies entirely within a region where replica symmetry is not preserved, and that this line does not cross into a region where both q_{11} and q_2 are non-zero†. These features weaken the prospective link between an ergodic system and the spin-glass model of Sherrington and Kirkpatrick when the latter is considered in a one-step replica-symmetry breaking scheme, wherein both qs are necessary within the single spin-glass phase. Indeed, even though the spins may here favourably influence their own mutual interactions, there do not appear to be any regions (figure 2) in which replica-symmetry breaking can be avoided while either choosing n according to the balancing

† This would appear to hold also for smaller values of $\tilde{J}$, as would the existence of re-entrant Greek RSB only below $n \simeq 0.3$ and the occurrence of a cusp in the Roman AT line at $n = 1$, $\beta = 1$.

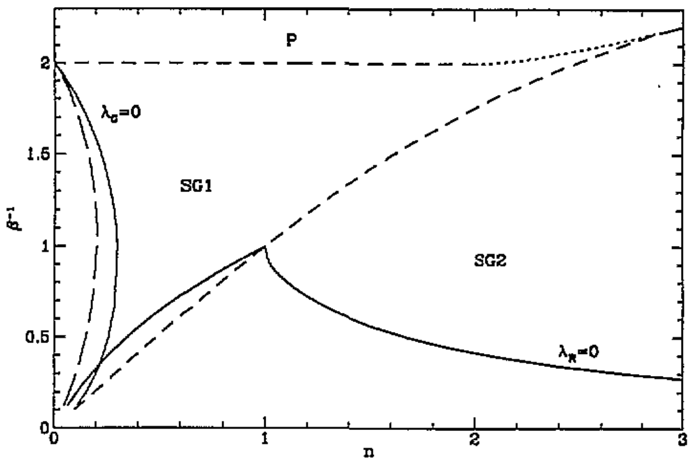


Figure 2. The phase diagram for the quenched, random bias model is shown for $B_0 = 0$, $\tilde{B} = 1$, $\tilde{J} = 3$. The short dashed curves denote second-order phase boundaries; the first-order part of the P–SG1 boundary is dotted where it is distinct from the P–SG2 boundary, and where the two are coincident, the boundary is draw in the continuing full line. The cusped contiguous curve denotes the Roman AT line while that lying below $n \simeq 0.3$ is the Greek AT line. The long-dashed curve is the line on which the spin and coupling free energies match, with larger coupling free energies being manifested towards greater values of n.

condition (4.4), or having both qs finite with n lying in the range $(0, 1)$. These issues themselves do parallel the need for levels of RSB beyond the first step in the SK model within the interior of the spin-glass phase. Nevertheless, considering that the biases B_{ij} encourage couplings that resemble those of the SK model and that there are yet significant regions of each spin-glass phase where replica symmetry is not broken, it would appear that the diffusion of the couplings may help reduce the incidence of replica-symmetry breaking in the frozen phases. Despite this, it is not clear on the basis of the results available from this model, whether it is in fact possible to choose weight dynamics that lead to spin-glass ordering closely similar to that of the SK model on modest timescales without there automatically existing many non-equivalent macroscopic states on long timescales.

6. Computer simulations

In terms of its thermodynamical behaviour, the main distinction of the coupled dynamical model having quenched random biases from conventional spin-glass models is the existence of two distinct forms of spin-glass phase, which is pertinent to the analytic similarity of this model with the SK model with one-step RSB. We have therefore sought to confirm that the two order parameters q_2 and q_{11} (as defined by (3.7)) do indeed play distinct roles, and become non-zero at different temperatures.

However, there are a number of obstacles to this goal, which combine the difficulties inherent to the canonical model (cf Penney *et al* 1993) and those associated with disordered systems in general (Fischer and Hertz 1991). Central to the model itself is the need to allow the spin system to reach thermal equilibrium before the couplings are changed significantly, and that these themselves should be allowed to reach equilibrium. Simulating the model as a Monte Carlo dynamics within a Langevin dynamics therefore is a labour-intensive undertaking, which already limits the size of system that can be reliably simulated to rather modest proportions. Further, in evaluating $\langle S_i S_j \rangle$ as an importance-sampled time-average over the spin dynamics, the need for careful selection of the timescale of this average is emphasised by the wish to distinguish q_{11} from q_2. Implicit in the theory for $B_0 = h = 0$ is the assumption that the symmetry $H(\{S_i\}) = H(\{-S_i\})$ is spontaneously broken, i.e. that $\langle S_i \rangle$ can be non-zero. Thus, in evaluating the trace in Z_β (2.5), all spin configurations cannot be included, even in principle. Analytically this difficulty can be subdued by including some form of small external field, h_i or B_0, performing the analysis while retaining this field until only ultimately the limit h_i, $B_0 \to 0$ is taken. Such an approach is awkward in a simulation, so a careful choice of averaging timescale is needed, such that $\langle S_i \rangle$ is both representative of its value in thermal equilibrium (favouring protracted averages), but respects the spontaneously broken symmetry (which requires restraint in the averaging time). However, if $q_{11} = (1/N)\sum_i \overline{\langle S_i \rangle}^2$ is ever to be zero while $q_2 = (1/N)\sum_i \overline{\langle S_i \rangle^2}$ is finite, the dynamics of the couplings must allow the magnetizations of individual spins to reverse as these interactions evolve. This requires that the timescale of the weight average should itself be large.

These subtle and conflicting considerations have generally proven difficult to satisfy and therefore, in order to assist with the task, a number of compromises have been tolerated. Firstly, in order to discourage the spin system from freezing with finite net magnetization (which risks leading to an unwanted ferromagnetic bias in the couplings via (2.2)), an analytically benign term has been added to the spin Hamiltonian,

$$H' = -\sum_{i<j} J_{ij} S_i S_j + \frac{1}{N}\left\{\sum_i S_i\right\}^2 \qquad (6.1)$$

such that for $h = B_0 = 0$, this term has no effect on the theory, but is computationally useful in discouraging ordered spin freezing. Secondly, given that the theory requires no more explicit choice of the spin dynamics other than that they should have an equilibrium dynamics characterized by the Boltzmann distribution, it is advantageous to choose dynamics that are convenient practically. A single spin-flip Metropolis dynamics has the attraction of simplicity and conciseness, and therefore forms the core of the updates of the spin system. In addition, however, in order an attempt to accelerate equilibration near critical temperatures (be they those of the spin system in isolation or those of the composite system), we have included intermittent multiple-spin processes. These involve considering flipping small groups of spins—chosen at random, in the absence of a superior scheme that is concisely implemented (cf Wolff 1989)—according to the implied change in the spin Hamiltonian (6.1) and the Metropolis algorithm (Metropolis *et al* 1953).

In other respects the simulations are almost identical to those applied to the original formulation of the model (Penney *et al* 1993), although we have tried adjusting the equilibration timescales in order to cater better to the new variation of the theory. In the notation of the latter paper, the entire set of spins are updated $R_1 = 500$ times in order to allow thermal equilibration for the pertaining values of the couplings, before taking thermal averages of the products $S_i S_j$ over the following $R_2 = 1500$ spin-updates. (Multi-spin processes occur once in every ten spin updates, and involve clusters of three spins.) After each iteration of this entire series of moves, the couplings are adjusted in regular time-steps of $t \to t + 0.02\tau$, and after $R_3 = 500$ such increments, the next $R_4 = 1500$ updates are used to measure the order parameters assuming the system now to be in complete thermal equilibrium. These measurements are themselves averaged over twenty realizations of the quenched disorder, and the associated standard deviations are used to produce the error-bars shown in figure 1. The consumptive nature of the dynamics—involving $N(R_1+R_2)(R_3+R_4)$

elementary spin updates per measurement—limits us to considering systems of only $N = 25$ sites. In terms of the theoretical picture of the dynamics, the parameters $R_{1...4}$ need each only be large enough to reflect the thermal equilibrium properties of the spins and couplings (without being so large that no spontaneous symmetry breaking is observed), but they are otherwise unrelated to the timescale τ. Although the latter may be used to adjust the speed of the coupling dynamics, because we have insisted on the spins continually maintaining their thermal equilibrium as the couplings gradually evolve, τ does not alter the order of magnitude of this speed relative to the very much faster spin processes.

While it is clear that the results shown in figure 1 do not show perfect agreement with the theoretical predictions, there are a number of encouraging correspondences. Firstly, for each of the temperature ratios n simulated, after allowance is made for the modest system size, the order parameter q_2 shows fair quantitative agreement with theory, and seems to predict a paramagnetic to spin-glass transition close to that indicated by the analysis. It is also clear that the two order parameters q_{11} and q_2 are distinct measures of the spin freezing, with a finite value for q_2 not implying that q_{11} should also be finite. Moreover, although the quantitative evolution of the measured q_{11} does not agree with theory when it is predicted to be finite, there is a plausible agreement in the location of the transition from the regime of $q_{11} = 0$ to that of $q_{11} > 0$, again making due allowances for the limited size of the system. However, it would seem that with the dynamical parameters chosen, freezing of the spin system is more abrupt than in the theoretical picture. Nevertheless, it is probable that if larger systems could be simulated, and over longer timescales, the theory would be more convincingly vindicated, although the present attempts do highlight the great awkwardness of a system with two levels of frustration and disorder.

7. Conclusions

We have examined a spin model in which both spins and their mutual interactions can evolve, but on different timescales. Disorder is present both explicitly as quenched random parameters in the the stochastic interaction dynamics and by implication also in the dynamics of the spin moments. Identifying the ratio of the characteristic temperatures of the spin and coupling systems as a relevant quantity, comparison of theoretical predictions of behaviour of this model with one in which there is no externally imposed disorder, shows that both lead to varied phase transition character as this ratio n is changed. Both formulations show that the point $n = 2$ separates second-order transitions between paramagnetism and spin-glass ordering from those of first order, but a new thermodynamic phase has been show to arise from the random biases on the weight dynamics. Moreover, the introduction of disorder into the model has been shown to extend the parallels of the original work with the replica method, and provides an alternative physical metaphor for the phenomenon of replica symmetry breaking, but suggests that rather different forms of coupling dynamics may be necessary if the pathological existence of many non-equivalent thermodynamic states is definitely to be avoided.

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Appendix

An outline of the derivation of the AT eigenvalues of the free-energy functional in the random-bias model will be given. Although the symmetries that maintain within all steps of the hierarchical replica symmetry breaking scheme invite the application of group-theoretic methods (e.g. Dorotheyev 1993), our less sophisticated methods follow those of de Almeida and Thouless (1978).

The procedure starts by considering small fluctuations of the order-parameters about replica-symmetric values:

$$q_a^{\alpha\beta} = q_2 + \varepsilon_a^{\alpha\beta} \quad \alpha < \beta \qquad q_{ab}^{\alpha\beta} = q_{11} + \eta_{ab}^{\alpha\beta} \quad a < b \tag{A.1}$$

whereafter the free-energy functional (3.4) is expanded up to second order in $\varepsilon_a^{\alpha\beta}$ and $\eta_{ab}^{\alpha\beta}$, a task that involves a great many combinatoric factors appropriate to the diverse topologies of spin contractions. By construction, the first-order terms vanish at the replica-symmetric saddle point. In terms of the following short hands:

$$J = \beta^2(\tilde{B} + \tilde{J}) \qquad B = \beta^2 \tilde{B} \tag{A.2}$$

there are three groups of matrix element; those relating solely to fluctuations about q_2,

$$\begin{aligned}
\langle \varepsilon_a^{\alpha\beta} | \mathcal{H} | \varepsilon_a^{\alpha\beta} \rangle &= K = J\left[1 - J\{1 - (q_2)^2\}\right] \\
\langle \varepsilon_a^{\alpha\beta} | \mathcal{H} | \varepsilon_a^{\alpha\gamma} \rangle &= L = J^2\left[(q_2)^2 - q_2\right] \\
\langle \varepsilon_a^{\alpha\beta} | \mathcal{H} | \varepsilon_a^{\gamma\delta} \rangle &= M = J^2\left[(q_2)^2 - r_4\right] \\
\langle \varepsilon_a^{\alpha\beta} | \mathcal{H} | \varepsilon_b^{\gamma\delta} \rangle &= N = J^2\left[(q_2)^2 - r_{22}\right]
\end{aligned} \tag{A.3}$$

those pertaining to deviations from q_{11},

$$\begin{aligned}
\langle \eta_{ab}^{\alpha\beta} | \mathcal{H} | \eta_{ab}^{\alpha\beta} \rangle &= P = B\left[1 - B\{1 - (q_{11})^2\}\right] \\
\langle \eta_{ab}^{\alpha\beta} | \mathcal{H} | \eta_{ab}^{\alpha\gamma} \rangle &= Q = B^2\left[(q_{11})^2 - q_2\right] \\
\langle \eta_{ab}^{\alpha\beta} | \mathcal{H} | \eta_{ab}^{\gamma\delta} \rangle &= R = B^2\left[(q_{11})^2 - r_{22}\right] \\
\langle \eta_{ab}^{\alpha\beta} | \mathcal{H} | \eta_{ac}^{\alpha\gamma} \rangle &= S = B^2\left[(q_{11})^2 - q_{11}\right] \\
\langle \eta_{ab}^{\alpha\beta} | \mathcal{H} | \eta_{ac}^{\gamma\delta} \rangle &= T = B^2\left[(q_{11})^2 - r_{211}\right] \\
\langle \eta_{ab}^{\alpha\beta} | \mathcal{H} | \eta_{cd}^{\gamma\delta} \rangle &= U = B^2\left[(q_{11})^2 - r_{1111}\right]
\end{aligned} \tag{A.4}$$

and matrix elements for mixed fluctuations

$$\begin{aligned}
\langle \varepsilon_a^{\alpha\beta} | \mathcal{H} | \eta_{ab}^{\alpha\gamma} \rangle &= V = JB\left[q_2 q_{11} - q_{11}\right] \\
\langle \varepsilon_a^{\alpha\beta} | \mathcal{H} | \eta_{ab}^{\gamma\delta} \rangle &= W = JB\left[q_2 q_{11} - r_{31}\right] \\
\langle \varepsilon_a^{\alpha\beta} | \mathcal{H} | \eta_{bc}^{\gamma\delta} \rangle &= X = JB\left[q_2 q_{11} - r_{211}\right]
\end{aligned} \tag{A.5}$$

where $r_4 = \langle \overline{\langle S \rangle^4} \rangle_B$, $r_{31} = \langle \overline{\langle S \rangle^3}\, \overline{\langle S \rangle} \rangle_B$ etc. (The labellings of these matrix elements are intended to represent generic examples of each type of element; all choices of indices with the same types of pairings between replicas in the bra and ket, consistent with the index constraints on $\varepsilon_a^{\alpha\beta}$ and $\eta_{ab}^{\alpha\beta}$, are to be understood.) That the Hessian $\mathcal{H}$ is symmetric implies that its eigenvectors must be orthogonal, a property that greatly assists in finding its eigenvalues.

In view of the observation made by de Almeida and Thouless (1978) that fluctuations that involve less than two distinct replicas appear always to be stable close to the replica-symmetric saddle point, we will confine attention to the replicon modes only. Given that there are two levels of replicas involved, replicon fluctuations come in two flavours, in analogy with q_2 and q_{11}. Firstly, there are modes that couple to two Greek replicas (θ, ν) within a single Roman replica (c), which are parameterized as follows:

$$\begin{aligned}
&\varepsilon_c^{\theta\nu} = a && \eta_{ac}^{\alpha\theta} = \eta_{ac}^{\alpha\nu} = e \\
&\varepsilon_c^{\theta\alpha} = \varepsilon_c^{\nu\alpha} = b \quad \alpha \neq \theta,\, \nu && \eta_{ac}^{\alpha\beta} = f \quad \beta \neq \theta,\, \nu \\
&\varepsilon_c^{\alpha\beta} = c \quad \alpha,\, \beta \neq \theta,\, \nu && \eta_{ab}^{\alpha\beta} = g \quad a, b \neq c \\
&\varepsilon_a^{\alpha\beta} = d \quad a \neq c\,.
\end{aligned} \tag{A.6}$$

Orthogonality of these vectors with those of the lower-order fluctuations (namely those that involve entirely homogeneous changes in $q_a^{\alpha\beta}$ and $q_{ab}^{\alpha\beta}$ and those that nominate a single special replica) implies that these vectors are limited to a one-dimensional subspace. Their eigenvalue may then be shown to be

$$\lambda_G = K - 2L + M \tag{A.7}$$

whose structure resembles the replicon mode of the SK model found by de Almeida and Thouless (1978), and that of the non-disordered coupled-dynamical model (Coolen *et al* 1993). The degeneracy of this eigenvalue is $\frac{1}{2}rn(n-3)$.

Secondly, modes coupling to two distinct Roman replicas (c, d) are also permitted, and are parameterized thus,

$$\begin{aligned}
&\varepsilon_c^{\theta\alpha} = \varepsilon_d^{\nu\alpha} = a \quad \alpha \neq \theta,\, \nu && \eta_{cd}^{\theta\nu} = d \\
&\varepsilon_c^{\alpha\beta} = \varepsilon_d^{\alpha\beta} = b \quad \alpha,\, \beta \neq \theta,\, \nu && \eta_{cd}^{\theta\alpha} = \eta_{cd}^{\alpha\nu} = e \quad \alpha \neq \nu,\, \theta \\
&\varepsilon_a^{\alpha\beta} = c \quad a \neq c,\, d && \eta_{ca}^{\theta\alpha} = \eta_{da}^{\nu\alpha} = f \quad a \neq c,\, d \\
& && \eta_{cd}^{\alpha\beta} = g \quad \alpha \neq \theta,\, \beta \neq \nu \\
& && \eta_{ab}^{\alpha\beta} = g \quad a,\, b \neq c,\, d\,.
\end{aligned} \tag{A.8}$$

Choosing the components a–g so as to ensure orthogonality with the zero- and one-replica fluctuations again leaves a single free parameter, and automatically means that this eigenvector is orthogonal to the Greek replicon mode. The resulting eigenvalue is then given by

$$\begin{aligned}
\lambda_R = P + &\frac{\{4(n-1)Q[1-n(r-1)]}{\{n^2r(r-1) - 2n(r-2) + 2\}} \\
&+ \frac{2[(n-1)^2R + 2(r-2)n(n-1)T + \frac{1}{2}(r-2)(r-3)n^2U]}{\{n^2r(r-1) - 2n(r-2) + 2\}} \\
&+ \frac{2(r-2)n[2-n(r-1)]S\}}{\{n^2r(r-1) - 2n(r-2) + 2\}}
\end{aligned} \tag{A.9}$$

whose degeneracy is $\frac{1}{2}rn(rn-n-2)$. The relevant limit of these eigenvalues, $r \to 0$, is readily effected.

References

de Almeida J R L and Thouless D J 1978 Stability of the SK solution of a spin glass model *J. Phys. A: Math. Gen.* **11** 983

Coolen A C C, Penney R W and Sherrington D 1993 Coupled dynamics of fast spins and slow interactions: an alternative perspective on replicas *Phys. Rev.* B **48** 16 116

de Dominicis C and Kondor I 1983 Eigenvalues of the stability matrix for the Parisi solution of the long-range spin glass *Phys. Rev.* B **27** 606

Dorotheyev A E 1993 Stability of the one-step replica-symmetry-broken phase in neural networks *J. Phys. A: Math. Gen.* **25** 5527

Edwards S F and Anderson P 1975 Theory of spin glasses *J. Phys. F: Met. Phys.* **5** 965

Fischer K H and Hertz J A 1991 *Spin Glasses* (Cambridge: Cambridge University Press)

Horner H 1984 Dynamic mean field theory of the SK-spin glass. I: Regularization by bond dynamics *Z. Phys.* B **57** 29

Kirkpatrick S and Sherrington D 1978 Infinite-ranged models of spin glasses *Phys. Rev.* B **17** 4384

Krauth W and Mézard M 1989 Storage capacity of memory with binary couplings *J. Physique* **50** 3057

Metropolis N, Rosenbluth A, Rosenbluth M, Teller A and Teller E 1953 Equation of state calculations by fast computing machines *J. Chem. Phys.* **21** 1087

Mézard M, Parisi G and Virasoro M 1987 *Spin Glass Theory and Beyond* (Singapore: World Scientific)

Monasson R and O'Kane D 1994 Domains of solutions and replica symmetry breaking in multilayer neural networks *Europhys. Lett.* submitted

Parisi G 1980a The order parameter for spin glasses: a function on the interval 0–1 *J. Phys. A: Math. Gen.* **13** 1101

—— 1980b Magnetic properties of spin glasses in a new mean field theory *J. Phys. A: Math. Gen.* **13** 1887

Penney R W, Coolen A C C and Sherrington D 1993 Coupled dynamics of fast spins and slow interactions in neural networks and spin systems *J. Phys. A: Math. Gen.* **26** 3681

Sherrington D and Kirkpatrick S 1975 Solvable model of a spin glass *Phys. Rev. Lett.* **35** 1792

Sompolinsky H 1981 Time-dependent order parameters in spin-glasses *Phys. Rev. Lett.* **47** 935–8

Viana L and Bray A 1985 Phase diagrams for dilute spin-glasses *J. Phys. C: Solid State Phys.* **18** 3037

Wolff U 1989 Collective monte carlo updating for spin systems *Phys. Rev. Lett.* **62** 361–4